TRINITY COLLEGE

Year 12 Mathematics Specialist 3/4
Test 6 2022
Weighting 6%

Calculator Assumed

Simple Harmonic Motion and Statistical Inference

STUDENT'S NAME Solutions [PRESSER]

DATE: Thursday 8 September **TIME:** 50 minutes **MARKS**: 50

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (10 marks)

Mr. Presser charges his phone each day and keeps accurate logs of his phone charging time. The time taken to charge the phone is normally distributed with mean $\mu = 55$ minutes and standard deviation $\sigma = 7$ minutes.

Mr. Presser randomly samples 64 phone charging times. Let $\bar{X}$ be the distribution of sample mean phone charging times for samples of size 64.

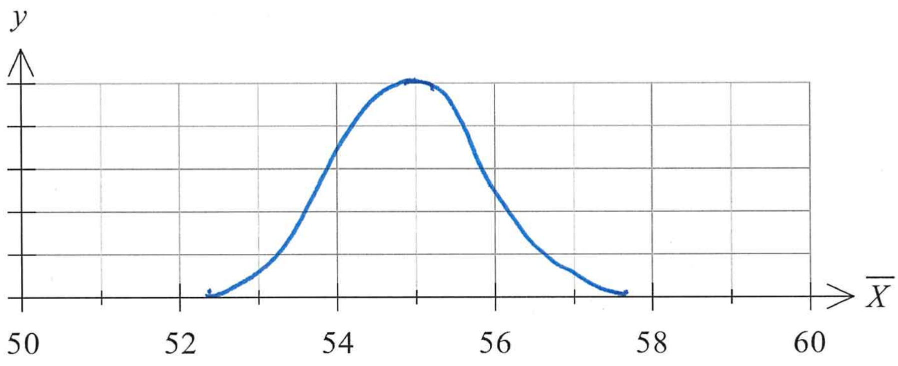
(a) Describe the distribution of $\bar{X}$. [3]

$n = 64 > 30$, so approximately normal.

$\bar{X} \sim N\left(55, \left(\frac{7}{\sqrt{64}}\right)^2\right) \sim N(55, 0.875^2)$

✓ $n > 30$ ~N
✓ μ
✓ variance

(b) Sketch the likely distribution of $\bar{X}$. [2]



✓ centre
✓ end pts

(c) Describe the change of the shape of distribution $\bar{X}$ if:

✓ decrease

(i) the sample size was to increase. [1]

s.d. of distribution would decrease, so the normal curve still centred on 55 but narrower.

(ii) the number of samples was to increase. [1]

The distribution of $\bar{x}$ would remain the same

✓ same

(d) Determine the probability that the total charge time is less than 55 hours. [3]

64

$\mu = 55$
$\sigma = 7$

parent

Using $\bar{x}$ from above

$\bar{x} = \frac{55 \times 60}{64}$

$= 51.5625$

$P(\bar{x} < 51.5625) = 0.0000427$

✓ sample mean
✓ correct prob statement
✓ ans

2. (9 marks)

The velocity-displacement equation of a body is $v^2 = \pi^2(9 - x^2)$.

(a) Without using trigonometric functions, show that the body is undergoing simple harmonic motion. [3]

$\Rightarrow \quad v^2 = 9\pi^2 - \pi^2 x^2$

taking d/dt of both sides ✓ $\frac{d}{dt}$

$\Rightarrow \quad \cancel{2} v \frac{dv}{dt} = -\cancel{2}\pi^2 x \frac{dx}{dt}$ ✓ expression for $\frac{dv}{dt}$ & $\frac{dx}{dt}$

$\Rightarrow \quad \frac{dv}{dt} = -\frac{\pi^2 x}{v} \cdot \frac{dx}{dt}$

$\Rightarrow \quad a = -\frac{\pi^2 x}{v} \cdot v$ ✓ $\frac{d^2x}{dt^2} = -k^2 x$

So $\frac{d^2x}{dt^2} = -\pi^2 x$

(b) Determine the

(i) period of the motion. [1]

Period $T = \frac{2\pi}{k} = \frac{2\pi}{\pi} = 2$ ✓ ans

(ii) maximum acceleration of the body. [2]

$v^2 = \pi^2(9 - x^2) \Rightarrow$ Amp $A = 3$ ✓ Amp

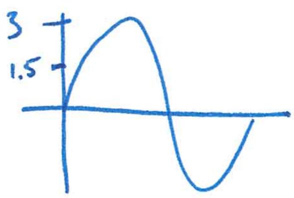
Now $a = \frac{d^2x}{dt^2} = -\pi^2 x$ ✓ ans

Max acc when x is max $\Rightarrow$ max acc $= 3\pi^2$

(iii) least time taken to move between the two points $x = 0$ and $x = 1.5$ [3]

$v^2 = \pi^2(9 - x^2) \Rightarrow x = 3\sin(\pi t + \alpha)$

No information about starting conditions, $\Rightarrow$ assume $\alpha = 0$



So $1.5 = 3\sin(\pi t)$ ✓ $x(t)$

$\frac{1}{2} = \sin \pi t$ ✓ eqn

$\Rightarrow \frac{\pi}{6} = \pi t$ ✓ t

$\Rightarrow t = \frac{1}{6}$

3. (14 marks)

The movement of a particle is modelled in terms of x, the displacement in cm from point P, and t, time in seconds.

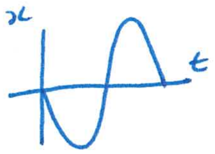
Given $\frac{d^2x}{dt^2} = -9x$, and that our particle was initially observed at P with a negative velocity and travels 15 cm in one cycle:

(a) Express x in terms of t. [3]

$$\frac{d^2x}{dt^2} = -(3)^2 x \qquad \text{SHM}$$ ✓ exp sin

$\Rightarrow \quad x = A\sin(3t + \alpha)$ ✓ Amp

But $-$ve velocity at $t=0$ and total distance of 15

$\Rightarrow$  $\Rightarrow$ Amp $= \frac{15}{4} = 3.75$ ✓ ans

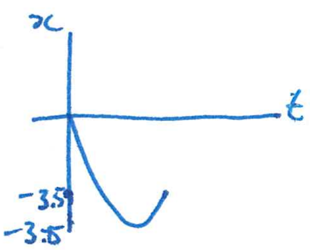
So $x = -\frac{15}{4}\sin(3t)$

(b) Calculate when the particle is first 1 cm away from P. [1]

$$-1 = \frac{-15}{4}\sin 3t$$

Solving $\Rightarrow$ $t = 0.08998$ sec ✓ ans

(c) Calculate when the particle has travelled a total distance of 4 cm. [2]



travels 4 cm when at $x = -3.5$ for the second time

$$-3.5 = -\frac{15}{4}\sin 3t$$ ✓ eqn

So $t = 0.646$ sec

✓ ans

OR $\quad 4 = \int_0^k \left|3\tfrac{15}{4}\cos 3t\right| dt$

Solving $k = 0.646$ sec

(d) Calculate how far the particle has travelled from $t = 0.4$ to $t = 0.85$, and hence the average speed over this time. [2]

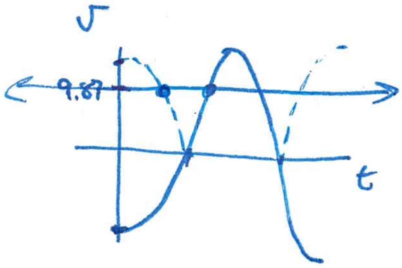
$$\text{dist} = \int_{0.4}^{0.85} \left| 3 \times 3.75 \cos 3t \right| dt$$

$$= 1.9135 \text{ cm}$$ ✓ dist

$$\text{So ave speed} = 1.9135 / 0.45$$

$$= 4.2523 \text{ cm/s}$$ ✓ ave speed

(e) Calculate the displacement of the particle when it first has an increasing ~~velocity~~ speed of 9.87 cm/s. [3]



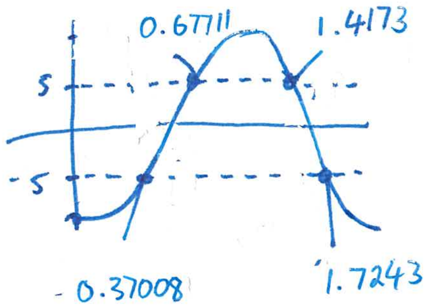
$$v = -3 \times 3.75 \cos 3t$$ ✓ v(t)

$$\text{So } 9.87 = \left| -3 \times 3.75 \cos 3t \right|$$ ✓ t

$$\text{soln } t = 0.1668,\ 0.8803$$

$$\text{Disp } x(0.8803) = -1.79955$$ ✓ x(t)

(f) Calculate the percentage of time the particle spends moving slower than 5 cm/s. [3]



✓ eqn

✓ t values

✓ %

$$\text{time between } \pm 5 \text{ cm/s} = 2 \times (0.6771 - 0.37008)$$

$$= 0.61406$$

$$\% = \frac{0.61406}{2\pi/3} \times 100\%$$

$$= 29.3\ \%$$

4. (17 marks)

A first sample of 50 pizzas has the weight of cheese recorded with a sample mean of 175.0 grams and a sample standard deviation of 13.4 grams.

(a) Based on the first sample, calculate the 95% confidence interval for the mean weight of cheese on a pizza. [3]

50

$\bar{x} = 175$

$s = 13.4$

95% C.I $\Rightarrow$ $P(-k < z < k) = 0.95$

$\Rightarrow$ $k = 1.96$ ✓ k

So 95% CI is $175 - 1.96 \times \frac{13.4}{\sqrt{50}} \le \mu \le 175 + 1.96 \times \frac{13.4}{\sqrt{50}}$ ✓ lower ✓ upper

$171.28\,g \le \mu \le 178.714\,g$

A second sample of 150 pizzas has the weight of cheese recorded and a 99% confidence interval is calculated. The lower limit of this interval is 167 grams, and the width of the interval is 6.3 grams.

(b) Determine the sample mean for the second sample. [2]

150

99% CI $\Rightarrow$ 167 — μ — 173.3

$\bar{y} = 167 + \frac{6.3}{2}$ ✓ lower + half

$= 170.15\,g$ ✓ ans

(c) Calculate, correct to 0.1 grams, the sample standard deviation for the sample of 150 pizzas. [3]

error $d = \frac{6.3}{2}$ ✓ error

$= 3.15$

and $3.15 = k\frac{s}{\sqrt{n}}$ ✓ eqn/ans

$\Rightarrow$ $3.15 = 2.576 \times \frac{s}{\sqrt{150}}$

$\Rightarrow$ $s = 14.977$

$\approx 15.0\,g$ ✓ ans 1d.p.

A third sample of n pizzas has the weight of cheese recorded and has a sample standard deviation of 3.8 grams.

(d) If the probability for the mean amount of cheese used differs from μ by less than 2 grams is 96%, calculate n, the number of pizzas that need to have their cheese weighed. [4]

$\bar{w} = ?$
$s = 3.8$

n

if n is large then $\bar{W} \sim N\left(\mu, \left(\frac{13.8}{\sqrt{n}}\right)^2\right)$

We require $P(|\bar{W} - \mu| < 2) = 0.96$

or $P(|z| < k) = 0.96$

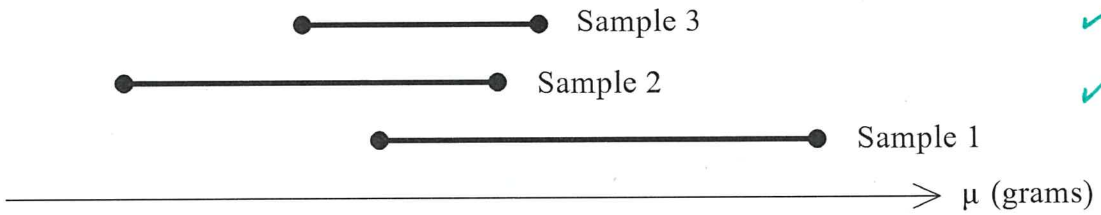
$\Rightarrow k = 2.0537$

Now $k = \dfrac{e}{s/\sqrt{n}}$ so $2.0537 = \dfrac{2}{13.8/\sqrt{n}}$

$\Rightarrow n = 200.8 \approx 201$

✓ $\bar{w}$
✓ k
✓ eqn
✓ n (integer)

The confidence intervals for each sample is shown below.



(e) A student claims that "*Sample 1 has a larger sample standard deviation that Sample 2 because the confidence interval is wider*". Comment on the validity of this claim. [3]

✓ s_1 and s_2

✓ CI %, sample size

✓ not valid

Sample 1: $s = 13.4$ Sample 2: $s = 15.09$
Sample 2 has a larger level of confidence and a larger sample size. The larger sample size for sample 2 is reducing the width of the confidence interval. Therefore, student statement is not valid.

(f) Which confidence interval is most likely to contain the value for μ? [2]

✓ random nature

✓ don't know which interval

Due to the inherent nature of random samples, and the fact that we do not know μ, we cannot determine which interval contains the true mean.